

Fractional excitation in one-dimensional two species fermionic superfluids

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We study one-dimensional two-species fermionic superfluids with order parameter twisted by an angle φ at the two ends. By solving the corresponding Bogoliubov-de-Gennes equation, we obtain the $U(1)$ soliton state which turns out to carry $\varphi S/(2\pi)$ spin, where S is the spin of single fermion. For Z_2 soliton with $\varphi = \pi$, the localized state carries only $1/2$ of a single fermion spin. To conserve the total spin of the system as an integral multiple of S , we demonstrate, the localized fractional spin $\varphi S/(2\pi)$ can be compensated by an opposite total spin $-\varphi S/(2\pi)$, which distributes uniformly in space and is thus undetectable locally in the thermodynamic limit.

I. INTRODUCTION

The imbalanced superfluids have attracted attentions recently, which is basically a problem of the coexistence of superconductivity and magnetism. For the singlet pairing case, the order parameter is adjusted self-consistently in space to accommodate the excess spins. One of the possible ground state is the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state^{1,2}, where the order parameter is sinusoidally modulated in space, and the spins are aggregated around the nodes of order parameter. In quasi-one-dimensional (Q1D) superfluids, it ought to be easier to observe regular modulation of the order parameter, since the 1D system only allows the order parameter oscillating in one direction, and the orbital effect of magnetic field is also suppressed.

In practice, the Bechgaard salts, like $(\text{TMTSF})_2\text{ClO}_4$ and $(\text{TMTSF})_2\text{PF}_6$, are Q1D conductors which can be made superconducting at low temperature $T_c \approx 1 \text{ K}$ ³. These materials are argued to be triplet pairing superconductor^{4,5}, therefore may not show the FFLO state as had been proposed in Ref.6. Q1D superfluids may also be realized in cold fermionic system by strong trapping potentials. There are a number of theoretical studies conducted on the Q1D imbalanced superfluids in the cold atomic systems⁷⁻¹⁵. The coexistence of superconductivity and magnetism can be understood in the mean field (MF) level through the Bogoliubov-de-Gennes (BdG) equation. For 1D system, the low energy behavior is described by two branches of Dirac fermions coupled to the order parameter $\Delta(x)$ ^{13,16}. Without spin imbalance, the order parameter is uniform and can be chosen as real and positive Δ_0 . When the spin imbalance is present, the self-consistent solution to the BdG equation indicates $\Delta(x)$ in the ground state is still real, and oscillates between $-\Delta_0$ and Δ_0 to form Z_2 kinks to accommodate the excess spins. It is found that each kink can accommodate one spin¹³. Such kind of Z_2 topological structure of order parameter has also been studied as edge states in Q1D organic superconductors with different pairing symmetries recently in Ref. 17.

In principle the Q1D superfluids only contains fermions, however in the MF level the BdG equation can be viewed as a fermion-boson coupled system. The order parameter plays the role of Bose field without dynamics.

As well known, the generic model of Dirac fermion coupled with a real Bose field, has been firstly considered by Jackiw and Rebbi¹⁸, where they shew in the semiclassical approximation the Z_2 soliton excitation may carry $e/2$ charge. Later Su, Schrieffer and Heeger¹⁹ found that the organic conductor polyacetylene may be described by the electron-phonon coupled model, where the Bose field is the optical phonon describing the alternating displacement of ions. Unfortunately, the half charge as the hallmark of Z_2 soliton in principle can not be observed in polyacetylene, since there are always two branches of Dirac fermions, i.e., the so called fermion doubling problem. Recently, the effort to search half charge was made in Ref.20, where they proposed an experiment based upon the spin Hall insulator, where on each edge of the sample, only one pair of Dirac fermion appears, and the spiral magnetization can play the role of static boson field. When the sample is wide enough, the two branches of Dirac fermions are decoupled indeed to avoid the fermion doubling problem.

In polyacetylene, both the charge and spin number are conserved, correspondingly, the combination of the two solitons in different branch of Dirac fermions leads to spin-charge separated quasiparticles. While in Q1D superfluids, the charge conservation has already been broken by the MF treatment of fermion pairing, and only the spin number is conserved, therefore the soliton can only carry conserved spin numbers. Due to the fermion doubling, the spin number can be 1, 0, and -1. Another difference between polyacetylene and Q1D superfluids is that the alternating displacement of carbon ions in polyacetylene can only be real, which means the corresponding soliton can only belong to Z_2 class. Generally, in the Q1D superfluids the order parameter $\Delta(x)$ can take complex values, therefore $\Delta(x)$ may be condensed in arbitrary different phases at the two ends, which may spoil the particle-hole symmetry. We call these solitons the *complex $U(1)$ soliton*, which may be related to the topological excitation in two dimensional topological matter, such as fractional quantum Hall effect²¹ and weak coupling p -wave pairing superconductor²².

Inspired by recent experimental and theoretical progress^{20,34} In this paper we shall consider these complex soliton states in Q1D superfluids, which may be realized by embedding the two ends of a Q1D supercon-

ductor into two different superconducting reservoirs with twisted phase φ . If $\Delta(x)$ varies very slowly in space, the spin number accumulated at the kink in the ground state should be $\varphi/(2\pi)$ according to Goldstone and Wilczek²³. Later Mackenzie and Wilczek provided another computation of this fractional quantum number²⁴, which is based upon the exact solution for some special forms of $\Delta(x)$. However both the calculations do not provide the origin of this fractional quantum number. Note that people usually consider the conserved total number of quasi-particle as “charge”, while in our case it corresponds to “spins”. These two cases are actually the same problem within the framework of BdG equation. Therefore in the following, we will abuse the terminology for a while, namely, the term “charge” and “spin” have the same meaning in the following. Anyway we shall avoid this misusing of language as possible as we can. Let’s scrutinize two problems. The first one is that the basic charge unit in condensed matter is e , if there is fractional charge appearing, one needs to consider how to balance them with charges e . This problem is usually solved by considering soliton and antisoliton pairs, or multiple solitons excited spontaneously to obtain an integer number of charges by cancellation of fractional numbers. The second problem is more critical. Suppose one can tune the twisted phase φ continuously from the uniform state to a soliton state, in this process the charge number is a conserved quantity, therefore if the uniform state at the beginning has $\mathcal{N}e$ charges, after turning on the twisted phase, the total charge should be kept unchanged and still be $\mathcal{N}e$. It is then natural to ask where the fractional quantum number comes from. In this paper, we will compute the spatial distribution of charge(or spin) density $s(x)$ in the presence of complex soliton based upon the exact solution of the corresponding BdG problem. It reveals that $s(x)$ consists of two parts, one is homogeneous $s_u = (\mathcal{N} + c)/L$, and the other is spatially dependent $s_l(x)$ which vanishes exponentially at the two ends, where $c = \varphi/(2\pi)$, and $\int dx s_l(x) = -c$. Clearly, the total particle number $\mathcal{N} = \int dx s(x)$ is unchanged by the twisted boundary condition. However, if one can locally detect the quasi-particle number, one may find a localized charge distribution $s_l(x)$, which is actually the so-called fractional charge. It is also interesting to note that the fractional charge in 1D system may also be adopted to interpret the fractional quantum Hall effect via the idea of edge soliton in Ref.21, though strictly speaking, the BdG theory in the MF level is not suitable to describe the 1D interacting fermions, and one needs the chiral Luttinger liquid²⁵ instead.

This paper is organized as following. In section II, we give a brief introduction of the Hamiltonian, and notations used in this paper. In section III we consider the single soliton excitation, including its energy and the effect of finite momentum cutoff in III.A and III.B, and the spin distribution in III. C. In section IV, a brief conclusion is given.

II. HAMILTONIAN

We consider 1D non-relativistic Fermi gas with attractive interaction, whose low energy behavior is controlled by the quasi-particles near the two Fermi points. There are four branches of relevant quasiparticles, two right moving fermions $\hat{R}_\sigma(x)$ and two left moving ones \hat{L}_σ with spin indexes $\sigma = \uparrow, \downarrow$. In the BCS theory, pairing takes place between $\hat{R}_\sigma(x)$ and $\hat{L}_{-\sigma}(x)$. The corresponding MF Hamiltonian reads

$$\hat{H} = \int dx [\hat{\mathcal{H}}_1(x) + \hat{\mathcal{H}}_2(x)] + \int dx \frac{|\Delta(x)|^2}{g} \quad (1)$$

where $g(>0)$ is the attractive interaction. The Hamiltonian density $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ read

$$\begin{aligned} \hat{\mathcal{H}}_1(x) &= -iv_F \hat{R}_\uparrow^\dagger \partial_x \hat{R}_\uparrow + iv_F \hat{L}_\downarrow^\dagger \partial_x \hat{L}_\downarrow \\ &\quad + \Delta(x) \hat{R}_\uparrow^\dagger \hat{L}_\downarrow^\dagger + \Delta^*(x) \hat{L}_\downarrow \hat{R}_\uparrow \\ \hat{\mathcal{H}}_2(x) &= iv_F \hat{L}_\uparrow^\dagger \partial_x \hat{L}_\uparrow - iv_F \hat{R}_\downarrow^\dagger \partial_x \hat{R}_\downarrow \\ &\quad + \Delta(x) \hat{L}_\uparrow^\dagger \hat{R}_\downarrow^\dagger + \Delta^*(x) \hat{R}_\downarrow \hat{L}_\uparrow \end{aligned} \quad (2)$$

where $\hbar = 1$ and v_F is the Fermi velocity. The variational principle leads to the self-consistent gap equation

$$\Delta(x) = -g[\langle \hat{L}_\downarrow \hat{R}_\uparrow \rangle + \langle \hat{R}_\downarrow \hat{L}_\uparrow \rangle] \quad (3)$$

We define two local spin operators for the two branches

$$\hat{s}_1(x) = \hat{R}_\uparrow^\dagger \hat{R}_\uparrow - \hat{L}_\downarrow^\dagger \hat{L}_\downarrow, \quad \hat{s}_2(x) = \hat{L}_\uparrow^\dagger \hat{L}_\uparrow - \hat{R}_\downarrow^\dagger \hat{R}_\downarrow, \quad (4)$$

whose integral over space are obviously conserved quantities.

$\hat{\mathcal{H}}_1$ can be written in a diagonalized form

$$\int dx \hat{\mathcal{H}}_1(x) = \sum_n \frac{\epsilon_n}{2} [\hat{d}_{1n}^\dagger \hat{d}_{1n} - \hat{d}_{1n} \hat{d}_{1n}^\dagger] \quad (5)$$

via substitution $\hat{R}_\uparrow(x) = \sum_n \hat{d}_{1n} u_n(x)$ and $\hat{L}_\downarrow^\dagger(x) = \sum_n \hat{d}_{1n} v_n(x)$ where $(u_n(x), v_n(x))^t \equiv \psi_n(x)$ satisfy the matrix differential equation,

$$[-iv_F \partial_x \sigma_3 + \Delta_1 \sigma_1 + \Delta_2 \sigma_2] \psi_n(x) = \epsilon_n \psi_n(x) \quad (6)$$

with $\Delta(x) = \Delta_1(x) - i\Delta_2(x)$. Note that $\psi_n(x)$ also satisfies

$$[iv_F \partial_x \sigma_3 + \Delta_1 \sigma_1 + \Delta_2 \sigma_2] \sigma_2 \psi_n^*(x) = -\epsilon_n \sigma_2 \psi_n^*(x) \quad (7)$$

therefore, if we substitute $\hat{L}_\uparrow(x) = -\sum_n \hat{d}_{2n} v_n^*(x)$ and $\hat{R}_\downarrow^\dagger(x) = \sum_n \hat{d}_{2n} u_n^*(x)$ into $\hat{\mathcal{H}}_2$, we have

$$\int dx \hat{\mathcal{H}}_2(x) = -\sum_n \frac{\epsilon_n}{2} [\hat{d}_{2n}^\dagger \hat{d}_{2n} - \hat{d}_{2n} \hat{d}_{2n}^\dagger]. \quad (8)$$

Since we consider a Q1D superfluid with two ends embedded in two different superfluid reservoirs, the phases

of order parameters at the two ends are consequently determined by the phases of the two reservoirs, respectively. To simulate this situation, we then impose a twisted boundary condition on the order parameter $\Delta(x)$,

$$\Delta(\pm \frac{L}{2}) = \Delta_0 e^{\pm i\varphi/2}, \quad (9)$$

where L is the system length and φ is twisted angle. The constraint Eq. (9) can also be implemented by the following boundary condition on wavefunctions

$$\lim_{x \rightarrow -\infty} \psi_n(x) = \lim_{x \rightarrow \infty} e^{-i\sigma_3 \varphi/2} \psi_n(x). \quad (10)$$

It is straightforward to verify the solutions of Eq. (7), $\sigma_2 \psi_n^*(x)$, also satisfy this boundary condition.

In case of $\varphi = 0$, it is the uniform superfluid phase with the uniform order parameter Δ_0 determined by

$$1 = \frac{g}{\pi} \int_0^\Lambda \frac{dk}{\omega(k)}, \quad (11)$$

where $\omega(k) = \sqrt{k^2 + \Delta_0^2}$ and Λ is the momentum cutoff. If we turn on the twisted angle φ of $\Delta(x)$ adiabatically, \hat{s}_1 and \hat{s}_2 are still conserved quantities. In this case there is a trivial solution with order parameter $\Delta(x) = \Delta_0 e^{i\varphi x/L}$, which obviously satisfies the boundary condition Eq. (9) and has a non-vanishing derivative with respect to x , i.e., $L\partial_x \Delta \neq 0$ at the boundary. By a gauge transformation $\hat{R}_\sigma(x) \rightarrow \hat{R}_\sigma(x) e^{i\varphi x/(2L)}$ and $\hat{L}_\sigma(x) \rightarrow \hat{L}_\sigma(x) e^{i\varphi x/(2L)}$ we go back to the uniform case. This kind of solution is not concerned to us in this paper, and we focus on the complex soliton states which have an additional requirement on order parameter $\lim_{x \rightarrow \pm\infty} L\partial_x \Delta = 0$.

Technically, the BdG equation Eq. (1), (3) for 1D superfluids is similar to those appearing in Gross-Neveu (GN) and Nambu-Jona-Lasinio (NJL) model, for which people have constructed various exact solutions base upon the inverse scattering methods. These solutions include the soliton, polaron, multisoliton, and so on in Refs. 26–29. There are also soliton lattice state³⁰. More recently, the spiral kink lattice state is given in Ref.31. In the following sections we apply these solutions to our problem with focus on single complex soliton.

Before going into the soliton solutions, we give the ground state energy of the system from Eq. (6) and Eq. (8),

$$E_g = - \sum_n |\epsilon_n| \quad (12)$$

which is not a simple summation over all negative eigenvalues, but also including the energy of fermion vacuum disturbed by inhomogeneous $\Delta(x)$.

III. SINGLE SOLITON EXCITATION

A. Order parameter and wavefunctions

In this section we follow the inverse scattering (IS) method to give the exact self-consistent solution of $\Delta(x)$ and the eigenpairs of $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ in the presence of inhomogeneous $\Delta(x)$. The mathematical description of IS method and its application to various 1D Dirac Fermion models can be found in Refs.26,27,29,32. In the following we give only the results, and for details one is referred to the references given above. Since the solution is relatively simple, one can verify the results directly.

For convenience a permutation of Pauli matrices, i.e., $\sigma_1 \rightarrow \sigma_3$, $\sigma_2 \rightarrow \sigma_1$ and $\sigma_3 \rightarrow \sigma_2$, is performed, so that the matrix differential operators in Eqs. (6) and (7), which are denoted by \mathbf{H}_1 and \mathbf{H}_2 respectively, read

$$\begin{aligned} \mathbf{H}_1 &= \begin{pmatrix} \Delta_1 & -\partial_x + \Delta_2 \\ \partial_x + \Delta_2 & -\Delta_1 \end{pmatrix}, \\ \mathbf{H}_2 &= \begin{pmatrix} \Delta_1 & \partial_x + \Delta_2 \\ -\partial_x + \Delta_2 & -\Delta_1 \end{pmatrix} \end{aligned} \quad (13)$$

where we set $v_F = 1$. The advantages of Eq. (13) is that both \mathbf{H}_1 and \mathbf{H}_2 are real, which implies the solutions have two-fold degeneracy and appear in complex conjugate pairs, though they may not be orthogonal to each other. The boundary condition Eq. (10) now reads,

$$\lim_{x \rightarrow -\infty} \psi_n(x) = e^{i\sigma_2 \varphi/2} \lim_{x \rightarrow \infty} \psi_n(x). \quad (14)$$

Note that $\mathbf{H}_2 = -\sigma_2 \mathbf{H}_1 \sigma_2$, therefore to obtain the spectra we can simply focus on \mathbf{H}_1 .

If $\varphi = 0$, the ground state is simply the uniform BCS state and there is no midgap state. If $\theta \neq 0$, we need at least one midgap state to match the boundary condition. From the IS method we know that $\Delta(x)$ which can minimize the free energy, should be reflectionless and has the form $\Delta(x) = \epsilon - i\kappa \tanh(\kappa x)$. To satisfy the boundary conduction Eq. (9), one need

$$\epsilon = \Delta_0 \cos \frac{\varphi}{2}, \quad \kappa = \Delta_0 \sin \frac{\varphi}{2}, \quad (15)$$

where Δ_0 is just the uniform solution satisfying Eq. (11). We restrict θ in the range $0 \leq \varphi < 2\pi$, therefore $\kappa \geq 0$. Then, \mathbf{H}_1 reads

$$\mathbf{H}_1 = \begin{pmatrix} \epsilon & -\partial_x + \kappa \tanh(\kappa x) \\ \partial_x + \kappa \tanh(\kappa x) & -\epsilon \end{pmatrix} \quad (16)$$

Its spectrum contains three parts, the negative scattering continuum with energy $\omega \leq -\Delta_0$, a single midgap state with energy ϵ , and the positive scattering continuum with energy $\omega \geq \Delta_0$. We recast the results to be more compact, therefore our result is different from those in Refs.27,29 by a rotation. For $\varphi = 0$, it is the uniform case, and for $\varphi = \pi$, it is the usual Z_2 soliton which has been extensively studied. In both cases, the system has

the particle-hole symmetry for an infinite system, i.e., a negative energy state can be transformed to be a positive energy state by a transformation $\psi \rightarrow \sigma_1 \psi$ if $\theta = 0$, and by $\psi \rightarrow \sigma_3 \psi$ if $\theta = \pi/2$. For a general value of φ other than 0 and π , it is not possible to find such a particle-hole transformation.

The complete eigenpairs to Hamiltonian Eq. (16) are listed below. The midgap state has eigenvalue ϵ , and the eigenfunction has a localized form

$$\psi^B(x, \kappa) = \sqrt{\frac{\kappa}{2}} \text{sech}(\kappa x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (17)$$

which vanishes at infinity. It is clear κ^{-1} can be thought as the characteristic width of this soliton. As we tune the twisted angle, this bound state of \mathbf{H}_1 is detached from the top of valence band, at the mean time, the bound state of \mathbf{H}_2 is lowered down from the bottom of conduction band as illustrated in Fig. 1. The scattering solutions

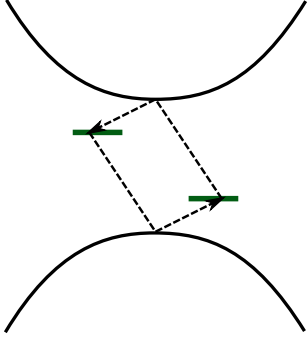


FIG. 1: Illustration of evolution of midgap states of both \mathbf{H}_1 and \mathbf{H}_2 branches.

with eigenvalue $\omega = \pm \sqrt{k^2 + \Delta_0^2}$ reads

$$\psi^S(x, k) \sim e^{ikx} \begin{pmatrix} \kappa \tanh(\kappa x) - ik \\ \omega - \epsilon \end{pmatrix} \quad (18)$$

In this form it is not clear that whether the two degenerate states $\psi^S(x, k)$ and $\psi^S(x, -k) = (\psi^S(x, k))^*$ are orthogonal to each other. To orthogonalize them, we define an inversion operator by

$$(\mathcal{P}\psi)(x) = \sigma_3 \psi(-x) \quad (19)$$

It is obvious that $\mathcal{P}\mathbf{H}_1\mathcal{P}^{-1} = \mathbf{H}_1$, then the eigenfunctions of \mathbf{H}_1 can be classified by parity $\mathcal{P}\psi_{\pm}(x) = \pm\psi_{\pm}(x)$. With the help of Eq. (18), we can construct orthonormal wavefunctions

$$\begin{aligned} \psi_+^S(x, k, \omega) &= N_+^S \begin{pmatrix} \kappa \tanh(\kappa x) \sin(kx) - k \cos(kx) \\ (\omega - \epsilon) \sin(kx) \end{pmatrix} \\ \psi_-^S(x, k, \omega) &= N_-^S \begin{pmatrix} \kappa \tanh(\kappa x) \cos(kx) + k \sin(kx) \\ (\omega - \epsilon) \cos(kx) \end{pmatrix} \end{aligned} \quad (20)$$

The normalization constants N_{\pm}^S are given in a box with length L

$$\begin{aligned} N_+^S &= \left[\omega(\omega - \epsilon)L + \frac{\sin(kL)}{k}(\omega\epsilon - \Delta_0^2) \right. \\ &\quad \left. - 2\kappa \sin^2 \frac{kL}{2} \tanh \frac{\kappa L}{2} \right]^{-1/2} \\ N_-^S &= \left[\omega(\omega - \epsilon)L - \frac{\sin(kL)}{k}(\omega\epsilon - \Delta_0^2) \right. \\ &\quad \left. - 2\kappa \cos^2 \frac{kL}{2} \tanh \frac{\kappa L}{2} \right]^{-1/2} \end{aligned} \quad (21)$$

At present, one can check immediately that Eqs. (17) and (20) are eigenfunctions of \mathbf{H}_1 with eigenvalues ϵ and ω , respectively. More importantly, all the eigenfunctions are orthogonal to each other, which is crucial to calculate the local spin distributions.

As for \mathbf{H}_2 ,

$$\mathbf{H}_2 = \begin{pmatrix} \epsilon & \partial_x + \kappa \tanh(\kappa x) \\ -\partial_x + \kappa \tanh(\kappa x) & -\epsilon \end{pmatrix}, \quad (22)$$

its eigenfunctions are just $\sigma_2 \psi^B(x, \kappa)$ and $\sigma_2 \psi_{\pm}^S(x, k, \omega)$ with an inverted band comparing to those of \mathbf{H}_1 .

B. Phase shift

The definition of phase shift for the present system is non-trivial due to the phase difference of $\Delta(x)$ between positive and negative infinity. Let's first check the asymptotic behaviors of \mathbf{H}_1 at infinity

$$\begin{aligned} \mathbf{H}_1 &\sim \begin{pmatrix} \epsilon & \kappa - \partial_x \\ \kappa + \partial_x & -\epsilon \end{pmatrix}, \text{ as } x \sim \infty, \\ \mathbf{H}_1 &\sim \begin{pmatrix} \epsilon & -\kappa - \partial_x \\ -\kappa + \partial_x & -\epsilon \end{pmatrix}, \text{ as } x \sim -\infty \end{aligned} \quad (23)$$

which differ from each other by a $\text{SU}(2)$ rotation $e^{i\sigma_2 \varphi/2}$. This difference does not allow us to define the phase shift δ in the usual way, i.e., $\lim_{x \rightarrow \infty} \psi(x) = e^{i\delta} \lim_{x \rightarrow -\infty} \psi(x)$. Similar problems also appear for the Z_2 kink in polyacetylene, which has been addressed as well as its physical consequences in Ref.33. To define a reasonable phase shift, one can eliminate the $\text{SU}(2)$ rotation of \mathbf{H}_1 at the two ends, which results in a definition the phase shift $\delta(k, \kappa)$ as

$$\lim_{x \rightarrow \infty} e^{i\sigma_2 \varphi/2} \psi^S(x, k, \omega) = e^{i\delta(k, \kappa)} \lim_{x \rightarrow -\infty} \psi^S(x, k, \omega) \quad (24)$$

Using Eq. (18) or Eq. (20) of scattering wavefunctions, the phase shift is computed as for positive k

$$\begin{aligned} \delta(k, \kappa) &= \pi - \text{atan} \frac{k}{\kappa} - \text{atan} \frac{\epsilon k}{|\omega| \kappa}, \text{ for conduction band,} \\ &\quad - \text{atan} \frac{k}{\kappa} + \text{atan} \frac{\epsilon k}{|\omega| \kappa}, \text{ for valence band.} \end{aligned} \quad (25)$$

which is also given in Ref.27.

C. Soliton energy

The total energy of the ground state in the presence of soliton is divergent as $\Lambda \rightarrow \infty$, just like in the uniform case. If one views the soliton state as a topological excitation above the uniform state, the excitation energy obtained by computing the energy difference between the soliton and uniform state is well defined and finite as $\Lambda \rightarrow \infty$. There are two sources to this energy difference, one is the readjustment of the fermion spectrum in the presence of soliton, the other is modification of the potential energy $\int dx |\Delta(x)|^2/g$.

To compute the excitation energy of soliton, we follow the method given in Refs.26 and 27. The basic idea is to quantize the pseudo momentum k by putting the system into a big enough box with length L which is required to be $L \gg \kappa^{-1}$, i.e., the system length is much larger than the characteristic width of the soliton. Thus, one can make a one-to-one correspondence between the spectra of soliton and uniform states. For the uniform case the momentum takes the value of $k_n = 2n\pi/L$ because there is no phase shift. When the twisted angle is turned on adiabatically, the pseudo-momentum \bar{k}_n is shifted from $k_n = 2n\pi/L$, and satisfies

$$\bar{k}_n^{(c,v)} L + \delta^{(c,v)}(\bar{k}_n^{(c,v)}, \kappa) = k_n L \quad (26)$$

where the phase shifts $\delta^{(c,v)}$ of the valence and conduction bands are defined in Eq. (25). The corresponding energy spectrum is also shifted slightly. The one can compute the energy difference between the soliton and uniform state, which turns out to be

$$\begin{aligned} \Delta E(\Lambda) &= \frac{2\omega_c}{\pi} \operatorname{atan} \frac{\kappa}{\Lambda} + \frac{2\epsilon}{\pi} \operatorname{atan} \frac{\Lambda\epsilon}{\kappa\omega_c} - |\epsilon| \\ \lim_{\Lambda \rightarrow \infty} \Delta E(\Lambda) &= \frac{2\kappa}{\pi} + \frac{2\epsilon}{\pi} \operatorname{atan} \frac{\epsilon}{\kappa} - |\epsilon| \end{aligned} \quad (27)$$

where $\omega_c = \sqrt{\Lambda^2 + \Delta_0^2}$. It is obvious $\Delta E = 0$ for $\varphi = 0$ or 2π , since it is just uniform state. For $\varphi = \pi$, $E_g = 2\Delta_0/\pi$ in the limit $\Lambda \rightarrow \infty$ which is well known for the Z_2 soliton. In condensed matter system or cold atomic gas, usually the energy cutoff can not be infinity, therefore the energy difference given in Eq. (27) is a function of cutoff Λ which decreases as Λ is increasing.

D. Fractional spin distributions

Using normalized orthogonal wavefunctions Eqs. (20) and (21), one can calculate the local spin density $s_T(x)$ of the system, which consists of two parts, $s_1(x)$ and $s_2(x)$

$$\begin{aligned} s_T(x) &= s_1(x) + s_2(x), \\ s_1(x) &= \langle 0 | \hat{R}_\uparrow^\dagger(x) \hat{R}_\uparrow(x) - \hat{L}_\downarrow^\dagger(x) \hat{L}_\downarrow(x) | 0 \rangle, \\ s_2(x) &= \langle 0 | \hat{L}_\uparrow^\dagger(x) \hat{L}_\uparrow(x) - \hat{R}_\downarrow^\dagger(x) \hat{R}_\downarrow(x) | 0 \rangle, \end{aligned} \quad (28)$$

where $|0\rangle$ is the ground state in the presence of soliton and $s_{1,2}$ belong to the two branches of $\hat{\mathcal{H}}_{1,2}$, respectively.

In fact for ordinary 1D spinful fermions as in our case, the emergent Dirac quasiparticles describing the low energy physics always appear in pairs. However, only one branch of Dirac fermion is also meaningful for some topological materials like two dimensional quantum spin Hall insulators, where the two branches Dirac fermions occurring as edge states are separated macroscopically in space²⁰, therefore can be treated individually. We

The operators \hat{R}_\uparrow and \hat{L}_\downarrow can be expanded in terms of the eigenfunctions of \mathbf{H}_1 . Taking the permutation of Pauli matrices in Sec. III A we obtain

$$s_1(x) = \sum_n \langle 0 | \hat{d}_{1n}^\dagger \hat{d}_{1n} | 0 \rangle |\psi_n(x)|^2 - \frac{1}{2} \sum_n |\psi_n(x)|^2 \quad (29)$$

where $\psi_n(x)$ is a short notation of those wavefunctions Eq. (17) and (20). The summation in the second term in the r.h.s. of Eq. (29) is over the full Hilbert space. It serves as a background full of spin-down particles, which originates from the particle-hole transformation for the spin-down operators. It is obviously a constant $(2N+1)/L$ in the uniform superfluids, where $2N+1$ states within the momentum cutoff are taken into account. In the presence of soliton this term still has a leading contribution $(2N+1)/L$, which is shown in the appendix A.

We now compute the first term in r.h.s. of Eq. (29) by assuming only the scattering states with negative energy are filled. The contribution of the midgap state can be computed straightforwardly to be $\kappa \operatorname{sech}^2(\kappa x)/2$ if it is filled. The spin density now reads

$$\begin{aligned} s_1(x) &= \sum_{\omega \leq -\Delta_0, n=1}^N [|\psi_+^S(x, \bar{k}_n, \omega)|^2 + |\psi_-^S(x, \bar{k}_n, \omega)|^2] \\ &\quad + |\psi_-^S(x, \bar{k}_0 = 0, -\Delta_0)|^2 - \frac{2N+1}{L} \end{aligned} \quad (30)$$

Note that the $\bar{k} = 0$ state in the negative band remains there when the twisted angle of $\Delta(x)$ is turned on, and it has no degeneracy unlike the other states with $\bar{k} \neq 0$ in the negative band.

Obviously, if one want to compute the total spin number for the vacuum of $\hat{\mathcal{H}}_1$, one gets $\int dx s_1(x) = 0$, or an integer if there are some thermal excitations, since $\psi_\pm^S(x, \bar{k}_n, \omega)$ and $\psi^B(x, \kappa)$ are normalized wavefunctions. This agrees with *the conservation law of the total spin* in the presence of the twisted angle φ of $\Delta(x)$. It means there is no possibility to get a fractional number by directly computing the integral of $s_1(x)$ in the way shown above. One may ask then where is the fractional excitations? We show our computation to this problem in the following. The key point is computing the local spin density $s_1(x)$ instead of its integral over space.

Substituting Eqs. (20) and (21) into Eq. (30) and considering only the allowed \bar{k}_n given in Eqs. (25) and (26), one finds that the spin density $s_1(x)$ can be written as the sum of two parts, the uniform s_{1u} and the spatially dependent one s_{1l} .

$$s_1(x) = s_{1u} + s_{1l}(x)$$

$$s_{1u} = \frac{c}{L}$$

$$s_{1l}(x) = -\frac{c\kappa}{2} \text{sech}^2(\kappa x) \quad (31)$$

where $c = -\varphi/(2\pi) = \theta/\pi$. The detailed calculation is left in appendix A.

Eq. (31) also implies $\int dx s_1(x) = 0$, which respects the spin conservation law. However if people can detect the local spin distribution in some way, it should be found that there is a fractional spin number accumulated around the origin ($x = 0$) with density profile described by $s_{1l}(x)$, while the uniform one s_{1u} is not detectable as $L \rightarrow \infty$. The fractional spin should be referred to $s_{1l}(x)$, not s_{1u} , although their integrals are both a fractional number c . This resolves the puzzles raised previously. Usually, people use the arguments of multi-soliton(antisoliton) or the fermion doubling to guarantee the spin conservation law. In our present situation, we show that *a single complex soliton excitation carrying a fraction number of spin can exist independently in a 1D fermionic system with only one branch of Dirac fermions.*

Similarly one can compute the spin density $s_2(x)$, which comes from the branch of \mathbf{H}_2 . Note that negative band of \mathbf{H}_2 consists of wavefunctions $\sigma_2 \psi^{(\pm)}(x, k_n, \omega)$ with $\omega \geq \Delta_0$, therefore

$$s_2(x) = \sum_{\omega > \Delta_0, n=1}^N \left[|\psi_+^S(x, \bar{k}_n, \omega)|^2 + |\psi_-^S(x, \bar{k}_n, \omega)|^2 \right] - \frac{2N+1}{L} \quad (32)$$

It is noticed that the $\bar{k} = 0$ state is missing for the positive band of \mathbf{H}_1 and becomes a midgap state, so does that in the negative band of \mathbf{H}_2 . Finally,

$$s_2(x) = s_{2u} + s_{2l}(x)$$

$$s_{2u} = -\frac{c}{L}$$

$$s_{2l} = -\frac{\kappa(1-c)}{2} \text{sech}^2(\kappa x) \quad (33)$$

Now let's consider the real ground state of Q1D superconductor in the presence of a single complex soliton, which always includes one midgap state of either $\hat{\mathcal{H}}_1$ branch or $\hat{\mathcal{H}}_2$ branch, besides the scattering states with negative energy. The total spin density is then zero $0 = s_1(x) + s_2(x) + \frac{\kappa}{2} \text{sech}^2(x)$, which also explains why the background spin distribution is a constant $(2N+1)/L$. If there are some extra quasiparticle excitations upon this ground state, which can only carry an integer number of spins, i.e., the fermion doubling smears the fractional number. To really observe the fraction excitations, one must get rid of the fermion doubling problem, possibly in some topological superconductor or insulators.

At last, we give a comment on this complex U(1) soliton for its ability to accommodate excess spins, which can be achieved by filling the positive midgap state $|\epsilon|$, therefore the energy cost per spin is $2\kappa/\pi + 2\epsilon \text{atan}(\epsilon/\kappa)/\pi$ (see

Eq. (27)). It is clear that $\theta = \pi/2$, i.e., Z_2 soliton with $\epsilon = 0$, is most energetically favorable as shown in Ref.13.

IV. CONCLUSION

We consider the complex soliton excitations in Q1D superconductors which has an order parameter with arbitrarily twisted phase angle at the two ends. Such kind of system may be realistic in Q1D organic superconductors, and the solitons discussed in this paper may be viewed as domain walls in an aggregation of Q1D superconductors, or in cold atomic two species Fermi gas for which it has been reported that the 1D imbalanced superfluid state were realized very recently³⁴. We give the exact self-consistent solution of a single complex soliton to the corresponding BdG equation. The usual Z_2 soliton is included as a simple case. The excitation energy of a single soliton is given for a finite energy cutoff. More interestingly for only one branch of Dirac fermions, we show in details the fractional excitation in this system by utilizing the exact wavefunctions of fermions in the presence of a complex soliton. It turns out the spin density actually consists of two parts, uniform and the spatial dependent one, which are compensated with each other(modulo 1) in the sense of integration over the whole space. In other words, there is a totally fractional number c of spins accumulated locally around the kink with $-c$ spins left evenly distributed to the background. This mechanism allows a single complex soliton excitation without breaking the spin conservation law, if people can get rid of the fermion doubling problem. Although there is no such report that only one pair of Dirac fermions appears in superconducting system, it is possible to realized such a fermionic system in the edge of 2D spin Hall insulators²⁰ where the fractional excitation is with respect to charge, not spin as in this paper, but the argument on fractional number given in this paper is essentially applicable to those systems. Furthermore in condensed matter system the charges are easy to be screened due to long ranged Coulomb interaction, while the spins, in other words, number differences of two species, may be free to screening effect, therefore might be easier to be detected.

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Appendix A: Derivation of spin density in space

We decompose the calculation for the spin density into three pieces, which are contributed from the negative band, midgap state, and positive band, respectively. It

will soon be clear that summation over these three pieces leads to a uniform background $2(2N+1)/L$, which should be subtracted when computing the spin density as shown in Eq. (29) and (30). To distinguish from the spin density defined previously, we denote these three pieces as $\tilde{s}_1(x)$, $\tilde{s}_m(x)$, and $\tilde{s}_2(x)$. The contribution of the midgap state is easy to evaluate

$$\tilde{s}_m(x) = \frac{\kappa}{2} \text{sech}^2(\kappa x) \quad (\text{A1})$$

The other two parts read

$$\begin{aligned} \tilde{s}_1(x) &= \sum_{\omega \leq -\Delta_0, n=1}^N [|\psi_+^S(x, \bar{k}_n, \omega)|^2 + |\psi_-^S(x, \bar{k}_n, \omega)|^2] \\ &\quad + |\psi_-^S(x, \bar{k}_0 = 0, -\Delta_0)|^2 \\ \tilde{s}_2(x) &= \sum_{\omega > \Delta_0, n=1}^N [|\psi_+^S(x, \bar{k}_n, \omega)|^2 + |\psi_-^S(x, \bar{k}_n, \omega)|^2] \end{aligned} \quad (\text{A2})$$

Note that when the twisted phase φ of $\Delta(x)$ is turned on, the state with $k = 0$ at the bottom of the positive band drops down and becomes the midgap state. This is why the summation in \tilde{s}_2 does not include $\bar{k} = 0$ term, which is actually taken into account by the midgap state in \tilde{s}_m .

In the following we give the detailed calculation for $\tilde{s}_1(x)$, and the calculation for \tilde{s}_2 is quite similar. The basic idea is expanding the spin density in terms of L^{-1} and extracting the leading order contributions by using the exact wavefunctions given in Eqs. (20) and (21). Therefore, those terms decaying exponentially fast as L tends to infinity are ignored, e.g., we take $\tanh(\kappa L) = 1$.

The state with $\bar{k} = 0$ has no degeneracy, and the corresponding wavefunction has odd parity. Its contribution to the spin density reads

$$|\psi_-^S(x, \bar{k}_0, -\Delta_0)|^2 = \frac{2\Delta_0(\Delta_0 + \epsilon) - \kappa^2 \text{sech}^2(\kappa x)}{2\Delta_0(\Delta_0 + \epsilon)L - 2\kappa} \quad (\text{A3})$$

For the states with $n \geq 1$, there is always a two-fold degeneracy for each \bar{k} . To compute their contribution to the spin density, we introduce the following notations for convenience,

$$\begin{aligned} \omega &= \sqrt{k^2 + \Delta_0^2} \\ f(x, k) &= \omega(\omega + \epsilon) - \frac{\kappa^2}{2} \text{sech}^2(\kappa x) \\ g(x, k) &= \left(\Delta_0^2 + \omega\epsilon - \frac{\kappa^2}{2} \text{sech}^2(\kappa x) \right) \cos(2kx) \\ &\quad + k\kappa \tanh(\kappa x) \sin(2kx) \\ A(k) &= \omega(\omega + \epsilon)L - \kappa \\ B(k) &= -\frac{\sin(kL)}{k}(\omega\epsilon + \Delta_0^2) + \kappa \cos(kL) \end{aligned} \quad (\text{A4})$$

Considering Eq. (26), one may get $\sin(\bar{k}^{(\pm)}L) = \sin(\delta^{(\pm)}(\bar{k}^{(\pm)}, \kappa))$. Further with the phase shift formula

Eq. (25), one can prove $\sin(\bar{k}L)/\bar{k}$ is finite for small $\bar{k} \neq 0$. Therefore, $A(k)$ is a quantity of order L , while $B(k)$ is only of order 1. Using Eq. (A4), one finds

$$\begin{aligned} &\sum_{n=1}^N [|\psi_+^S(x, \bar{k}_n, -\omega)|^2 + |\psi_-^S(x, \bar{k}_n, -\omega)|^2] \\ &= \sum_{n=1}^N \left[\frac{f(x, \bar{k}_n) - g(x, \bar{k}_n)}{A(\bar{k}_n) + B(\bar{k}_n)} + \frac{f(x, \bar{k}_n) + g(x, \bar{k}_n)}{A(\bar{k}_n) - B(\bar{k}_n)} \right] \\ &= \sum_{n=1}^N \frac{2A(\bar{k}_n)f(x, \bar{k}_n)}{A^2(\bar{k}_n) - B^2(\bar{k}_n)} + \sum_{n=1}^N \frac{2B(\bar{k}_n)g(x, \bar{k}_n)}{A^2(\bar{k}_n) - B^2(\bar{k}_n)} \end{aligned} \quad (\text{A5})$$

Note that $g(x, k)$ is an oscillating function of k and $B(\bar{k}_n)$ is finite, the second term in the equation above is proportional to an oscillating function multiplied by L^{-2} , which can be neglected safely. Therefore

$$\begin{aligned} &\sum_{n=1}^N [|\psi_+^S(x, \bar{k}_n, -\omega)|^2 + |\psi_-^S(x, \bar{k}_n, -\omega)|^2] \\ &\approx \sum_{n=1}^N \frac{2A(\bar{k}_n)\omega(\omega + \epsilon)}{A^2(\bar{k}_n) - B^2(\bar{k}_n)} - \sum_{n=1}^N \frac{A(\bar{k}_n)\kappa^2 \text{sech}^2(\kappa x)}{A^2(\bar{k}_n) - B^2(\bar{k}_n)} \end{aligned} \quad (\text{A6})$$

which has been divided into two parts, the uniform part and the spatially dependent part.

The uniform part can be computed as

$$\begin{aligned} &2 \sum_{n=1}^N \frac{A(\bar{k}_n)\omega(\omega + \epsilon)}{A^2(\bar{k}_n) - B^2(\bar{k}_n)} \\ &= \sum_{n=1}^N \frac{\omega(\omega + \epsilon)}{A(\bar{k}_n) - B(\bar{k}_n)} + \frac{\omega(\omega + \epsilon)}{A(\bar{k}_n) + B(\bar{k}_n)} \\ &= \sum_{n=1}^N \left[L - \frac{\kappa + B(\bar{k}_n)}{\omega(\omega + \epsilon)} \right]^{-1} + \left[L - \frac{\kappa - B(\bar{k}_n)}{\omega(\omega + \epsilon)} \right]^{-1} \\ &= \frac{2N}{L} + \frac{1}{L} \frac{\kappa}{\pi} \int_0^\infty dk \frac{1}{\omega(\omega + \epsilon)} + o(L^{-2}) \\ &= \frac{2N + c}{L} + o(L^{-2}) \end{aligned} \quad (\text{A7})$$

where

$$c = \frac{\kappa}{\pi} \int_0^\infty dk \frac{1}{\omega(\omega + \epsilon)} = \frac{\theta}{\pi} = -\frac{\varphi}{2\pi} \quad (\text{A8})$$

with $0 \leq \theta < \pi$. Here, we take the cutoff Λ/Δ_0 to be infinity.

Let's compute the spatial dependent part,

$$\begin{aligned} &\sum_n \frac{2A(\bar{k}_n)}{A(\bar{k}_n)^2 - B(\bar{k}_n)^2} \\ &= \sum_n \frac{1}{A(\bar{k}_n) - B(\bar{k}_n)} + \frac{1}{A(\bar{k}_n) + B(\bar{k}_n)} \\ &= \frac{1}{\pi} \int_0^\infty \frac{dk}{\omega(\omega + \epsilon)} + o(L^{-2}) \end{aligned}$$

$$= \frac{c}{\kappa} + o(L^{-2}) \quad (\text{A9})$$

Finally, taking the contribution from Eq. (A3), we get $\tilde{s}_1(x)$ in the limit $L \rightarrow \infty$

$$\tilde{s}_1(x) = \frac{2N+1+c}{L} - c\frac{\kappa}{2} \text{sech}^2(\kappa x). \quad (\text{A10})$$

On can calculate $\tilde{s}_2(x)$ in the same way, which is related to the \mathbf{H}_2 branch as stated in previous text. In the following we give the results directly, but please be careful that the state with $\bar{k}^{(+)} = 0$ drops into the gap and has no contribution to $\tilde{s}_2(x)$.

$$\tilde{s}_2(x) = \frac{2N+1-c}{L} - \frac{\kappa}{2}(1-c) \text{sech}^2(\kappa x) \quad (\text{A11})$$

Taking all the states in the Hilbert space into account, one obtains a uniform back ground

$$\tilde{s}_1(x) + \tilde{s}_2(x) + \tilde{s}_m(x) = \frac{2(2N+1)}{L} \quad (\text{A12})$$

which should be subtracted when computing the spin density $s_{1,2}(x)$ as shown in Sec. III D.

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